# Signed spectral Turań type theorems 

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#### Abstract

A signed graph $\Sigma=(G, \sigma)$ is a graph where the function $\sigma$ assigns either 1 or -1 to each edge of the simple graph $G$. The adjacency matrix of $\Sigma$, denoted by $A(\Sigma)$, is defined canonically. In a recent paper, Wang et al. extended the spectral bounds of Hoffman and Cvetković for the chromatic number of signed graphs. They proposed an open problem related to the balanced clique number and the largest eigenvalue of a signed graph. We solve a strengthened version of this open problem. As a byproduct, we give alternate proofs for some of the known classical bounds for the least eigenvalues of unsigned graphs. We extend the Turán's inequality for the signed graphs. Besides, we study the Bollobás and Nikiforov conjecture for the signed graphs and show that the conjecture need not be true for the signed graphs. Nevertheless, the conjecture holds for signed graphs under some assumptions. Finally, we study some of the relationships between the number of signed walks and the largest eigenvalue of a signed graph.


Keywords. Signed graph, Eigenvalue, Balanced clique number, Edge bipartiteness, Signed walk.

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## 1 Introduction

A signed graph $\Sigma$ is a pair $(G, \sigma)$, where $G=(V, E)$ is an undirected graph, called the underlying unsigned graph, and $\sigma: E \rightarrow\{-1,+1\}$ is the sign function. Let $e_{i j}$ denote the edge between the vertices $v_{i}$ and $v_{j}$ of $G$. The adjacency matrix of a signed graph $\Sigma=(G, \sigma)$ is a symmetric matrix, denoted by $A(\Sigma)$ and its $(i, j)$-th entry is defined as follows:

$$
a_{i j}= \begin{cases}\sigma\left(e_{i j}\right) & \text { if } v_{i} \sim v_{j} \\ 0 & \text { otherwise }\end{cases}
$$

[^0]Let $\lambda_{1}(\Sigma) \geq \lambda_{2}(\Sigma) \geq \cdots \geq \lambda_{n}(\Sigma)$ be the eigenvalues of $A(\Sigma)$. The spectrum of $A(\Sigma)$ is the multiset of eigenvalues of $A(\Sigma)$. The spectrum and the spectral radius of $\Sigma$ are the spectrum and the spectral radius of $A(\Sigma)$ and denoted by $\operatorname{spec}(\Sigma)$ and $\rho(\Sigma)$, respectively. Let $(G, 1)$ and $(G,-1)$ denote the signed graphs with all the edge signs are equal to 1 and -1 , respectively. For more details about the notion of signed graphs, we refer to [8, 17, 18].

The sign of a cycle (with some orientation) $C=v_{1} v_{2} \ldots v_{l} v_{1}$, denoted by $\sigma(C)$, is defined as the product of the signs of its edges, that is

$$
\sigma(C)=\sigma\left(e_{12}\right) \sigma\left(e_{23}\right) \cdots \sigma\left(e_{(l-1) l}\right) \sigma\left(e_{l 1}\right),
$$

where $e_{i j}$ is the edge between the vertices $v_{i}$ and $v_{j}$. A cycle $C$ is neutral if $\sigma(C)=1$, and a signed graph is balanced if all its cycles are neutral. A function from the vertex set of $G$ to the set $\{1,-1\}$ is called a switching function. Two signed graphs $\Sigma_{1}=\left(G, \sigma_{1}\right)$ and $\Sigma_{2}=\left(G, \sigma_{2}\right)$ are switching equivalent, written as $\Sigma_{1} \sim \Sigma_{2}$, if there is a switching function $\eta: V \rightarrow\{1,-1\}$ such that

$$
\sigma_{2}\left(e_{i j}\right)=\eta\left(v_{i}\right)^{-1} \sigma_{1}\left(e_{i j}\right) \eta\left(v_{j}\right) .
$$

The switching equivalence of two signed graphs can be defined in the following equivalent way: Two signed graphs $\Sigma_{1}=\left(G, \sigma_{1}\right)$ and $\Sigma_{2}=\left(G, \sigma_{2}\right)$ are switching equivalent, if there exists a diagonal matrix $D_{\eta}$ with diagonal entries from $\{1,-1\}$ such that

$$
\begin{equation*}
A\left(\Sigma_{2}\right)=D_{\eta}^{-1} A\left(\Sigma_{1}\right) D_{\eta} . \tag{1.1}
\end{equation*}
$$

Switching equivalence preserves connectivity and balance.
A complete subgraph of an unsigned graph $G$ is a called a clique in $G$. The clique number of $G$, denoted by $\omega(G)$, is the number of vertices in a largest clique in $G$. Let $S_{n}^{+}=\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}: \sum x_{i}=1, x_{i} \geq 0\right.$ for $\left.i=1, \ldots, n\right\}$ and $S_{n}^{ \pm}=\left\{\left(x_{1}, \ldots, x_{n}\right) \in\right.$ $\left.\mathbb{R}^{n}: \sum_{i=1}^{n}\left|x_{i}\right|=1\right\}$.

In the seminal paper [10], Motzkin and Straus proved the following result.
Theorem 1.1 ([10, Theorem 1]). Let $G$ be a graph on $n$ vertices with the clique number $\omega(G)$. If $A(G)$ is the adjacency matrix $G$, then

$$
\max _{x \in S_{n}^{+}} x^{T} A(G) x=\frac{\omega(G)-1}{\omega(G)} .
$$

Using this result, Nikiforov came up with a proof of the following theorem, which was conjectured by Edwards and Elphick [5].

Theorem 1.2 ([11, Theorem 2.1]). Let $G$ be a graph with $n$ vertices and $m$ edges. Let $\lambda_{1}(G)$ be the largest eigenvalue of $A(G)$, and $\omega(G)$ be the clique number of $G$. Then

$$
\lambda_{1}^{2}(G) \leq 2 m \frac{\omega(G)-1}{\omega(G)}
$$

The balanced clique number of a signed graph $\Sigma$, denoted by $\omega_{b}(\Sigma)$, is the maximum number of vertices in a balanced complete subgraph. The MS-index of a signed graph $\Sigma=(G, \sigma)$, denoted by $\mu(\Sigma)$, is defined as follows:

$$
\mu(\Sigma)=\max _{x \in S_{n}^{ \pm}} \sum_{i \sim j} \sigma\left(e_{i j}\right) x_{i} x_{j}=\max _{x \in S_{n}^{ \pm}} \frac{x^{T} A(\Sigma) x}{2} .
$$

In [16], Wang et al. extended the Motzkin-Straus theorem for the signed graphs .
Theorem 1.3 ([16, Theorem 5]). Let $\Sigma=(G, \sigma)$ be a signed graph with balanced clique number $\omega_{b}(\Sigma)$. Then

$$
\mu(\Sigma)=\frac{1}{2}\left(\frac{\omega_{b}(\Sigma)-1}{\omega_{b}(\Sigma)}\right) .
$$

As a consequence of the above theorem, Wang et al. extended Cvetković's theorem for the signed graphs.

Theorem 1.4 ([16, Proposition 5]). Let $\Sigma=(G, \sigma)$ be a signed graph with $n$ vertices, and balanced clique number $\omega_{b}(\Sigma)$. Let $\lambda_{1}(\Sigma)$ be the largest eigenvalue of $A(\Sigma)$. Then

$$
\lambda_{1}(\Sigma) \leq n\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)
$$

In the same paper, Wang et al. explained the difficulties in extending Nikiforov's theorem (Theorem (1.2) for the signed graphs and mentioned it as an open problem. One of the main objectives of this article is to give a proof of a strengthened version of this open problem for the signed graph (see Theorem 3.3). We give an alternate proof for Theorem 1.4 too. As a byproduct, we derive a spectral lower bound for the edge bipartiteness of a simple graph and deduce the upper bounds for the least eigenvalues of unsigned graphs obtained by Constantine [4]. The edge bipartiteness of an unsigned graph $G$ is the least number of edges whose deletion yields a bipartite graph [6]. These results are given in Section 3.

Let $K_{n}$ denote the complete graph on $n$ vertices. In [3], Bollobás and Nikiforov proposed the following conjecture. If $G$ is a $K_{r+1}$-free graph on at least $r+1$ vertices with $m$ edges, and $\lambda_{1}(G) \geq \lambda_{2}(G) \geq \cdots \geq \lambda_{n}(G)$ are the eigenvalues of $A(G)$, then $\lambda_{1}(G)^{2}+\lambda_{2}(G)^{2} \leq 2 m\left(\frac{r-1}{r}\right)$. Recently Lin et al. [9] confirmed this conjecture for the triangle free graphs. We show that the signed graph version of this conjecture need not be true for the signed graphs even for the triangle free graphs. We prove the signed analogue of this conjecture for the triangle free graphs whose largest eigenvalue is the spectral radius. This is done in Section 4 .

Given an unsigned graph $G$, a $k$-walk is a sequence of vertices $v_{1}, \ldots, v_{k}$ of $G$ such that $v_{i}$ is adjacent to $v_{i+1}$ for all $i=1, \ldots, k-1$. The length of a walk is the number of edges it has, counting repeated edges as many times as they appear. So the length of a $k$-walk is $k-1$. Let $w_{r}(G)$ be the number of $r$-walks in $G$. Note that, if $G$ is an unsigned graph, then $w_{r}(G)=e^{T} A^{r-1}(G) e$, where $e=(1,1, \ldots, 1)^{T}$. Nikiforov proved the following result connecting the number of walks in $G$, chromatic number of $G$ and the spectral radius of the adjacency matrix $A(G)$ of $G$.

Theorem 1.5 ([12, Theorem 5 \& Theorem 14]). Let $G$ be a graph with $n$ vertices and $m$ edges. Let $A(G)$ be its adjacency matrix and $\omega(G)$ be its clique number. Let $\lambda_{1}(G) \geq \lambda_{2}(G) \geq$ $\cdots \geq \lambda_{n}(G)$ be the eigenvalues of the adjacency matrix of $G$. Then for $r>0$ and odd $q>0$,

$$
\frac{w_{q+r}(G)}{w_{q}(G)} \leq \lambda_{1}^{r}(G) \leq w_{r}(G) \frac{\omega(G)-1}{\omega(G)} .
$$

In Section 5, we extend the above result for the signed graphs. As a corollary to this result, we obtain Theorem 1.4.

## 2 Preliminaries

In this section, we collect the needed matrix theoretic preliminaries about doubly stochastic matrices. For more details on related knowledge, we refer the reader to [2, 19]. A nonnegative square matrix is doubly stochastic if the sum of the entries in every row and every column is 1 , and it is doubly substochastic if the sum of the entries in every row and every column is at most 1 .

For a vector $x \in \mathbb{R}^{n}$, let $x^{\downarrow}$ denote the vector obtained by rearranging the coordinates of $x$ in the decreasing order. Given two vectors $x, y \in \mathbb{R}^{n}$, we say that $x$ is weakly majorised by $y$, denote by $x \prec_{w} y$ if

$$
\sum_{j=1}^{k} x_{j}^{\downarrow} \leq \sum_{j=1}^{k} y_{j}^{\downarrow} \text { for all } 1 \leq k \leq n
$$

If $x \prec_{w} y$ and $\sum_{i=1}^{n} x_{i}^{\downarrow}=\sum_{i=1}^{n} y_{i}^{\downarrow}$, then we say that $x$ is majorized by $y$ and denote it by $x \prec y$.

Lemma 2.1 ([19, Theorem 3.9]). Let $x, y \in \mathbb{R}_{+}^{n}$. Then $x \prec_{w} y$ if and only if there exists $a$ doubly substochastic matrix $A$ such that $x=A y$.

For $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$ and $1 \leq p<\infty$, define $\|x\|_{p}=\left\{\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right\}^{\frac{1}{p}}$.
Theorem $2.2\left(\left[9\right.\right.$, Theorem 2.1]). Let $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right), y=\left(y_{1}, y_{2}, \ldots, y_{n}\right) \in \mathbb{R}_{+}^{n}$, such that $x_{i}$ and $y_{i}$ are in non-increasing order. If $y \prec_{w} x$, then $\|y\|_{p} \leq\|x\|_{p}$ for any real number $p>1$, where equality holds if and only if $x=y$.

Theorem 2.3 (The Interlacing Theorem). Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times m}$ be a principal submatrix of $A$. Suppose $A$ has eigenvalues $\lambda_{1} \leq \lambda_{2} \leq \cdots \leq \lambda_{n}$ and $B$ has eigenvalues $\beta_{1} \leq \beta_{2} \leq \cdots \leq \beta_{m}$. Then

$$
\lambda_{k} \leq \beta_{k} \leq \lambda_{n-m+k} \quad \text { for } k=1, \ldots, m
$$

## 3 Eigenvalues and Balanced clique number

In this section, we first prove an open problem posed by Wang et al. [16], and as a corollary, we obtain some of the known classical bounds for the least eigenvalues of the adjacency matrices of unsigned graphs. Then we prove Turán's theorem for signed graphs. Thereupon we prove a signed version one of Nikiforov's bound for the largest eigenvalue of unsigned graph. As a consequence of this we give an different proof for a result of Stanić. On a final note we derive an alternate proof of Motzkin-Strauss theorem for signed graphs.

For a signed graph $\Sigma$, the frustration index, denoted by $\epsilon(\Sigma)$, is the minimum number of edges to be deleted such that the resultant signed graph is balanced [1].

Let $\Sigma=(G, \sigma)$ be a signed graph. An edge $e$ of $\Sigma$ is positive (resp. negative) if $\sigma(e)=1$ (resp. $\sigma(e)=-1$ ). For a signed graph $\Sigma$, let $m^{+}(\Sigma)$ denote the number of positive edges in $\Sigma$, and $m^{-}(\Sigma)$ denote the number of negative edges in $\Sigma$.
Lemma 3.1. Let $\Sigma$ be a signed graph with $n$ vertices, $m$ edges and frustration index $\epsilon(\Sigma)$. If $\Sigma$ and $\Sigma^{\prime}$ are switching equivalent, then $m^{+}\left(\Sigma^{\prime}\right) \leq m-\epsilon(\Sigma)$.
Proof. Let $H$ be the signed subgraph of $\Sigma^{\prime}$ consisting of all the positive edges of $\Sigma^{\prime}$. Then $H$ is balanced, and hence $\epsilon\left(\Sigma^{\prime}\right) \leq m^{-}\left(\Sigma^{\prime}\right)$. Thus $m^{+}\left(\Sigma^{\prime}\right) \leq m-\epsilon(\Sigma)$, as $\epsilon(\Sigma)=\epsilon\left(\Sigma^{\prime}\right)$.

Essentially Lemma 3.1 says that the number of positive edges in any signed graph in the whole switching equivalence class of $\Sigma=(G, \sigma)$ is bounded above by $m-\epsilon(\Sigma)$.

Next, we prove a lemma which is crucial in the proofs of the main theorems.
Lemma 3.2. Let $\Sigma=(G, \sigma)$ be a signed graph with the balanced clique number $\omega_{b}(\Sigma)$. If $\Sigma$ and $\Sigma^{\prime}$ are switching equivalent, then $\omega_{b}(\Sigma)=\omega_{b}\left(\Sigma^{\prime}\right)$.

Proof. Since switching in signed graphs preserves the balance of cycles, the balance of cliques must also be preserved.

In [16], Wang et al. proved the Motzkin-Strauss theorem and Wilf's theorem for signed graphs. In the same paper extending Theorem 1.2 for the signed graphs is mentioned as an open problem. Next, we prove a stronger version of this open problem.

Theorem 3.3. Let $\Sigma=(G, \sigma)$ be a signed graph with $m$ edges. If $\lambda_{1}(\Sigma)$ is the largest eigenvalue of the adjacency matrix $A(\Sigma)$, then

$$
\lambda_{1}^{2}(\Sigma) \leq 2(m-\epsilon(\Sigma))\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)
$$

where $\omega_{b}(\Sigma)$ and $\epsilon(\Sigma)$ are the balanced clique number and frustration index of $\Sigma$, respectively.
Proof. Let $x=\left(x_{1}, \ldots, x_{n}\right)$ be an eigenvector corresponding to the eigenvalue $\lambda_{1}(\Sigma)$ with $\|x\|_{2}=1$. Define $y=\left(\left|x_{1}\right|, \ldots,\left|x_{n}\right|\right)$. Consider the switching function $\eta: V(G) \rightarrow\{1,-1\}$ defined as follows:

$$
\eta\left(v_{i}\right)=\left\{\begin{array}{l}
\operatorname{sgn} x_{i} \quad \text { if } \quad x_{i} \neq 0 \\
1 \quad \text { otherwise }
\end{array}\right.
$$

Now, switch the signed graph $\Sigma=(G, \sigma)$ to $\Sigma^{\prime}=\left(G, \sigma^{\prime}\right)$ using the switching function $\eta$ as defined above. Then $A\left(\Sigma^{\prime}\right)=D_{\eta}^{-1} A(\Sigma) D_{\eta}$. Note that $\lambda_{1}(\Sigma)=\lambda_{1}\left(\Sigma^{\prime}\right)$ and $\omega_{b}(\Sigma)=\omega_{b}\left(\Sigma^{\prime}\right)$. Now,

$$
\begin{aligned}
\lambda_{1}(\Sigma) & =x^{T} A(\Sigma) x \\
& =y^{T} D_{\eta}^{-1} A(\Sigma) D_{\eta} y \\
& =y^{T} A\left(\Sigma^{\prime}\right) y \\
& =2 \sum_{i \sim j} \sigma_{i j}^{\prime} y_{i} y_{j} \quad\left[\sigma_{i j}^{\prime} \text { is the }(i, j)^{t h} \text { entry of } A\left(\Sigma^{\prime}\right)\right] \\
& =\sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j}-\sum_{i \sim j, \sigma_{i j}^{\prime}=-1} 2 y_{i} y_{j} .
\end{aligned}
$$

As all the entries of the vector $y$ are non-negative, we have

$$
\begin{equation*}
\lambda_{1}(\Sigma) \leq \sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j} \tag{3.1}
\end{equation*}
$$

and hence

$$
\lambda_{1}(\Sigma)^{2} \leq\left(\sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j}\right)^{2}
$$

Let $\Sigma_{+}^{\prime}$ be the signed subgraph of $\Sigma^{\prime}$ consisting of all the positive edges of $\Sigma^{\prime}$. Let $m^{+}$be the number of edges in $\Sigma_{+}^{\prime}$, and $\omega\left(\Sigma_{+}^{\prime}\right)$ be the clique number of $\Sigma_{+}^{\prime}$. By the Cauchy-Schwartz inequality, we have

$$
\left(\sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j}\right)^{2} \leq 2 m^{+} \sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i}^{2} y_{j}^{2}
$$

Since all the edges of the signed subgraph $\Sigma_{+}^{\prime}$ are positive and the vector $y$ is a unit eigenvector (that is, $\|y\|_{2}=1$ ), by Theorem 1.1, we have

$$
\sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i}^{2} y_{j}^{2} \leq\left(1-\frac{1}{\omega\left(\Sigma_{+}^{\prime}\right)}\right)
$$

Thus

$$
\lambda_{1}(\Sigma)^{2} \leq 2 m^{+}\left(1-\frac{1}{\omega\left(\Sigma_{+}^{\prime}\right)}\right)
$$

Since $\Sigma_{+}^{\prime}$ is the subgraph of $\Sigma^{\prime}$ containing all positive edges, by Lemma 3.1, we have $m^{+} \leq$ $m-\epsilon(\Sigma)$ and $\omega\left(\Sigma_{+}^{\prime}\right) \leq \omega_{b}(\Sigma)$. Thus

$$
\lambda_{1}^{2}(\Sigma) \leq 2(m-\epsilon(\Sigma))\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)
$$

Next, we prove some of the known classical bounds on the least eigenvalues of the adjacency matrices of unsigned graphs using Theorem 3.3. The edge bipartiteness of an unsigned graph $G$ is the least number of edges whose deletion yields a bipartite graph [6]. The following is a result of Favaron et al. [7], and the result was stated in a different form. We rewrite the bound in terms of edge bipartiteness.

Corollary 3.4 ([7, Theorem 2.17]). Let $G$ be a graph with $n$ vertices and $m$ edges. Let $A(G)$ be its adjacency matrix and $\omega(G)$ be its clique number. Let $\epsilon_{b}(G)$ denote the edge bipartiteness of the graph $G$. Let $\lambda_{n}(G)$ be the least eigenvalue of $A(G)$. Then

$$
\lambda_{n}^{2}(G) \leq m-\epsilon_{b}(G) .
$$

Proof. Let $\Sigma=(G,-1)$ be the signed graph with all negative edges on $G$. Then, by Theorem 3.3, we get

$$
\lambda_{1}^{2}(\Sigma)=\lambda_{n}^{2}(G) \leq 2(m-\epsilon(\Sigma))\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)
$$

In the signed graph $\Sigma=(G,-1)$, triangles are not balanced, so $\omega_{b}(\Sigma)=2$. Note that $\Sigma$ is balanced if and only if $G$ is bipartite, thus $\epsilon(\Sigma)=\epsilon_{b}(G)$. Substituting these in the expression above, gives the desired result.

From Corollary 3.4, we can also prove two other classical bounds for $\lambda_{n}(G)$ for a graph $G$.

Corollary 3.5 ([4, Proposition 2]). Let $G$ be a graph with $n$ vertices and $m$ edges. Let $A(G)$ be its adjacency matrix and $\omega(G)$ be its clique number. Let $\lambda_{n}(G)$ be the least eigenvalue of $A(G)$.

1. If $n=2 k$, then $\left|\lambda_{n}(G)\right| \leq k$.
2. If $n=2 k+1$, then $\left|\lambda_{n}(G)\right| \leq \sqrt{k(k+1)}$.

Proof. From Corollary 3.4, we have

$$
\lambda_{n}^{2}(G) \leq m-\epsilon_{b}(G)
$$

By the definition of $\epsilon_{b}(G), m-\epsilon_{b}(G)$ is the number of edges in a maximal bipartite subgraph of $G$. Let $V_{1} \cup V_{2}$ be a bipartition of the vertex set of a maximal bipartite subgraph obtained by removing $\epsilon_{b}(G)$ edges. Then

$$
\left|\lambda_{n}(G)\right| \leq \sqrt{m-\epsilon_{b}(G)} \leq \sqrt{\left|V_{1}\right|\left|V_{2}\right|} .
$$

We have the following two cases:

1. If $\left|V_{1}\right|+\left|V_{2}\right|=n=2 k$, then by AM-GM inequality, $\sqrt{\left|V_{1}\right|\left|V_{2}\right|} \leq k$.
2. If $\left|V_{1}\right|+\left|V_{2}\right|=n=2 k+1$, then let $\left|V_{1}\right|=k+a,\left|V_{2}\right|=k+1-a$, where $a$ is an integer. Then $\sqrt{\left|V_{1}\right|\left|V_{2}\right|}=\sqrt{(k+a)(k+1-a)}=\sqrt{k(k+1)+a-a^{2}} \leq \sqrt{k(k+1)}$, as $a$ is an integer.

Next we recall the classical Turań inequality for the unsigned graphs.
Theorem 3.6. Let $G$ be a graph on $n$ vertices and $m$ edges with clique number $\omega$. Then

$$
m \leq \frac{n^{2}}{2}\left(1-\frac{1}{\omega(\Sigma)}\right)
$$

The following theorem extends Turán's inequality for the signed graphs in terms of the balanced clique number and frustration index.

Theorem 3.7. Let $\Sigma=(G, \sigma)$ be a signed graph with $n$ vertices, $m$ edges, balanced clique number $\omega_{b}(\Sigma)$ and the frustration index $\epsilon(\Sigma)$. Then

$$
m \leq \epsilon(\Sigma)+\frac{n^{2}}{2}\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)
$$

Proof. Since $\epsilon(\Sigma)$ is the frustration index, so there exists a set, say $S$, of $\epsilon(\Sigma)$ edges such that the signed graph $\Sigma \backslash S$ is balanced. Let $\eta$ be the switching function which switches $\Sigma \backslash S$ to a signed graph with all positive edges. Let $\omega$ be the clique number of $\Sigma \backslash S$. By Lemma 3.1 and Turán's theorem for unsigned graphs, we have

$$
m^{+}(\Sigma \backslash S) \leq m-\epsilon(\Sigma) \leq \frac{n^{2}}{2}\left(1-\frac{1}{\omega}\right)
$$

As $\omega \leq \omega_{b}(\Sigma)$, we get

$$
m \leq \epsilon(\Sigma)+\frac{n^{2}}{2}\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)
$$

For a real number $x$, let $\lfloor x\rfloor$ denote the greatest integer less than or equal $x$. For a simple graph $G$ on $m$ edges, Stanley [15] established that $\rho(A(G)) \leq \frac{1}{2}(-1+\sqrt{1+8 m})$. Later Nikiforov [13] proved the following improved bound

$$
\rho(A(G)) \leq \sqrt{2\left(1-\left\lfloor\frac{1}{2}+\sqrt{2 m+\frac{1}{4}}\right\rfloor^{-1}\right) m} .
$$

We extend Nikiforov's bound for the signed graphs in the next theorem. The proof idea is similar to that of Nikiforov [13].

Theorem 3.8. Let $\Sigma=(G, \sigma)$ be a signed graph with $n$ vertices, $m$ edges, balanced clique number $\omega_{b}(\Sigma)$ and frustration index $\epsilon(\Sigma)$. Then

$$
\lambda_{1}(\Sigma) \leq \sqrt{2(m-\epsilon(\Sigma))\left(1-\left\lfloor\frac{1}{2}+\sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}}\right\rfloor^{-1}\right)},
$$

where $\lambda_{1}(\Sigma)$ is the largest eigenvalue of $A(\Sigma)$.

Proof. Let $\Omega$ be a balanced complete subgraph of $\Sigma$ with $\omega_{b}(\Sigma)$ vertices. Let $\eta^{\prime}$ be a switching function on $\Omega$ which switches all the edges of $\Omega$ to positive. Define the switching function $\eta$ on $\Sigma$ as follows:

$$
\eta\left(v_{i}\right)=\left\{\begin{array}{l}
\eta^{\prime}\left(v_{i}\right) \quad \text { if } \quad v_{i} \in \Omega \\
1 \quad \text { otherwise }
\end{array}\right.
$$

Let $\Sigma^{\prime}$ be the signed graph obtained from $\Sigma$ by applying the switching $\eta$. Then, by Lemma 3.1 ,

$$
\binom{\omega_{b}(\Sigma)}{2} \leq m^{+}\left(\Sigma^{\prime}\right) \leq m-\epsilon(\Sigma) .
$$

By solving the above expression for $\omega_{b}(\Sigma)$, we get

$$
\omega_{b}(\Sigma) \leq \frac{1}{2}+\sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}} .
$$

The result follows by noting that $\omega_{b}(\Sigma)$ is a positive number and hence

$$
\omega_{b}(\Sigma) \leq\left\lfloor\frac{1}{2}+\sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}}\right\rfloor .
$$

Substituting this version in Theorem 3.3 proves the result.
As a corollary of above the theorem we get the following result of Stanić.
Corollary 3.9 ([14, Theorem 4.2]). Let $\Sigma=(G, \sigma)$ be a signed graph with $n$ vertices, $m$ edges and frustration index $\epsilon(\Sigma)$. Then

$$
\lambda_{1}(\Sigma) \leq \sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}}-\frac{1}{2}
$$

where $\lambda_{1}(\Sigma)$ is the largest eigenvalue of $A(\Sigma)$.
Proof. From the proof of the above theorem, we have

$$
\omega_{b}(\Sigma) \leq \frac{1}{2}+\sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}} .
$$

Substituting the above bound in Theorem 3.3, we get

$$
\begin{aligned}
\lambda_{1}(\Sigma) & \leq \sqrt{2(m-\epsilon(\Sigma))\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)} \\
& \leq \sqrt{2(m-\epsilon(\Sigma)) \frac{-\frac{1}{2}+\sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}}}{\frac{1}{2}+\sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}}}} \\
& =-\frac{1}{2}+\sqrt{2(m-\epsilon(\Sigma))+\frac{1}{4}} .
\end{aligned}
$$

Let us recall the definition of the MS-index of a signed graph $\Sigma=(G, \sigma): \mu(\Sigma)=$ $\max _{x \in S_{n}^{ \pm}} \sum_{i \sim j} \sigma\left(e_{i j}\right) x_{i} x_{j}=\max _{x \in S_{n}^{ \pm}} \frac{x^{T} A(\Sigma) x}{2}$. Next, we give an alternate proof for Theorem 1.3 .

Theorem 3.10 ([16, Theorem 5]). Let $\Sigma=(G, \sigma)$ be a signed graph with balanced clique number $\omega_{b}(\Sigma)$. Then

$$
\mu(\Sigma)=\frac{1}{2}\left(\frac{\omega_{b}(\Sigma)-1}{\omega_{b}(\Sigma)}\right) .
$$

Proof. Let $x=\left(x_{1}, \ldots, x_{n}\right) \in S_{n}^{ \pm}$be a vector such that $\mu(\Sigma)=\sum_{i \sim j} x_{i} x_{j}$. Define $y=$ $\left(\left|x_{1}\right|, \ldots,\left|x_{n}\right|\right)$. Consider the switching function $\eta: V(G) \rightarrow\{1,-1\}$ defined as follows:

$$
\eta\left(v_{i}\right)=\left\{\begin{array}{l}
\operatorname{sgn} x_{i} \text { if } x_{i} \neq 0 \\
1 \text { otherwise }
\end{array}\right.
$$

Now, switch the signed graph $\Sigma$ to $\Sigma^{\prime}$ by using the switching function $\eta$ as defined above. Then $A\left(\Sigma^{\prime}\right)=D_{\eta}^{-1} A(\Sigma) D_{\eta}$. Note that $\mu(\Sigma)=\mu\left(\Sigma^{\prime}\right)$ and $\omega_{b}(\Sigma)=\omega_{b}\left(\Sigma^{\prime}\right)$. Now,

$$
\begin{aligned}
2 \mu(\Sigma) & =x^{T} A(\Sigma) x \\
& =y^{T} D_{\eta}^{-1} A(\Sigma) D_{\eta} y \\
& =y^{T} A\left(\Sigma^{\prime}\right) y \\
& =2 \sum_{i \sim j} \sigma_{i j}^{\prime} y_{i} y_{j} \quad\left[\sigma_{i j}^{\prime} \text { is the }(i, j)^{t h} \text { entry of } A\left(\Sigma^{\prime}\right)\right] \\
& =\sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j}-\sum_{i \sim j, \sigma_{i j}^{\prime}=-1} 2 y_{i} y_{j} .
\end{aligned}
$$

As all the entries of the vector $y$ are non-negative, we have

$$
2 \mu(\Sigma) \leq \sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j}
$$

Let $\Sigma_{+}^{\prime}$ be the signed subgraph of $\Sigma^{\prime}$ consisting of all the positive edges of $\Sigma^{\prime}$. Let $m^{+}$be the number of edges in $\Sigma_{+}^{\prime}$, and $\omega\left(\Sigma_{+}^{\prime}\right)$ be the clique number of $\Sigma_{+}^{\prime}$. By Theorem 1.1, we have

$$
\sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j} \leq 2 \mu\left(\Sigma_{+}^{\prime}\right)=\frac{\omega\left(\Sigma_{+}^{\prime}\right)-1}{\omega\left(\Sigma_{+}^{\prime}\right)} .
$$

Since $\omega\left(\Sigma_{+}^{\prime}\right) \leq \omega_{b}(\Sigma)$, we have

$$
\mu(\Sigma) \leq \frac{1}{2} \frac{\omega_{b}(\Sigma)-1}{\omega_{b}(\Sigma)}
$$

Now, let $\Omega$ be a balanced complete subgraph of $\Sigma$ with $\omega_{b}(\Sigma)$ vertices. Let $\eta^{\prime}$ be a switching function on $\Omega$ which switches all the edges of $\Omega$ to positive. Define the switching function $\eta$ on $\Sigma$ as follows:

$$
\eta\left(v_{i}\right)=\left\{\begin{array}{l}
\eta^{\prime}\left(v_{i}\right) \quad \text { if } \quad v_{i} \in \Omega \\
1 \quad \text { otherwise }
\end{array}\right.
$$

Let $\Sigma^{\prime \prime}$ be the signed graph obtained from $\Sigma$ by applying the switching function $\eta$. Let $z=\left(z_{1}, z_{2}, \ldots, z_{n}\right)^{T}$ be the vector given by

$$
z_{i}=\left\{\begin{array}{l}
\frac{1}{\omega_{b}(\Sigma)} \text { if } v_{i} \in \Omega \\
0 \text { otherwise }
\end{array}\right.
$$

Then, we have

$$
z^{T} A\left(\Sigma^{\prime}\right) z=\frac{\omega_{b}(\Sigma)-1}{\omega_{b}(\Sigma)} \leq 2 \mu\left(\Sigma^{\prime}\right)=2 \mu(\Sigma)
$$

and thus we have the result.

## 4 Bollobás and Nikiforov's conjecture: Signed version

In this section, we prove Lin et al.'s result for the signed graph with an additional restriction, and we also give a counterexample for the result when the restriction is dropped. This example also shows that the conjecture posed by Nikiforov and Bollobás does not hold for signed graphs in general. Then we derive a bound for the largest eigenvalue of a signed graph in terms of the number of edges and the number of triangles in it.

In [3], Bollobás and Nikiforov proposed the following conjecture.
Conjecture 4.1. If $G$ is a $K_{r+1}$-free graph of order at least $r+1$ with $m$ edges, and $\lambda_{1} \geq$ $\lambda_{2} \geq \cdots \geq \lambda_{n}$ are the eigenvalues of $A(G)$, then $\lambda_{1}^{2}+\lambda_{2}^{2} \leq \frac{r-1}{r} 2 m$.

Recently Lin et al. [9] confirmed this conjecture for the triangle-free graphs.
Theorem 4.1 ([9, Theorem 1.2]). Let $G$ be a triangle-free graph on $n$ vertices and $m$ edges with $n \geq 3$. Then $\lambda_{1}^{2}+\lambda_{2}^{2} \leq m$.

The following example shows that the above theorem, as well as the conjecture of Bollobás and Nikiforov, need not be true for signed graphs. Let $C_{5}$ denote the cycle on 5 vertices with $V\left(C_{5}\right)=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right\}$. Consider the signed graph $\Sigma=\left(C_{5}, \sigma\right)$ with $\sigma$ assigns -1 to the edge between the vertices $v_{1}$ and $v_{2}$, and 1 to the remaining edges. Then

$$
A(\Sigma)=\left[\begin{array}{ccccc}
0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{array}\right]
$$

The spectrum of $A(\Sigma)$ is $\{1.618,1.618,-0.618,-0.618,-2\}$. It is easy to see that $\lambda_{1}^{2}(\Sigma)+$ $\lambda_{2}^{2}(\Sigma)=5.2358>5$. Since $r=2$ and $m=5$, so $\frac{r-1}{r} 2 m=5$. Thus Conjecture 4.1 need not be true for signed graphs.

Nevertheless, we have the following result for the signed graph.
Theorem 4.2. Let $\Sigma$ be a signed graph with $n$ vertices and $m$ edges with $n \geq 3$, and $\Sigma$ has no balanced triangles. If $\lambda_{1}(\Sigma) \geq \lambda_{2}(\Sigma) \geq \cdots \geq \lambda_{n-1}(\Sigma) \geq \lambda_{n}(\Sigma)$ are the eigenvalues of $A(\Sigma)$ and $\lambda_{1}(\Sigma) \geq\left|\lambda_{n}(\Sigma)\right|$, then

$$
\lambda_{1}^{2}(\Sigma)+\lambda_{2}^{2}(\Sigma) \leq m .
$$

We need the following result in the proof of the above theorem.
Lemma 4.3. Let $\Sigma=(G, \sigma)$ be a signed graph on $n$ vertices with $n \geq 3$, and $\Sigma$ has no balanced triangles. Let $\lambda_{1}(\Sigma) \geq \lambda_{2}(\Sigma) \geq \cdots \geq \lambda_{n}(\Sigma)$ be the eigenvalues of $A(\Sigma)$. Then

$$
\lambda_{2}(\Sigma) \geq 0
$$

Proof. Since $\Sigma$ does not contain balanced triangles, either $\Sigma$ is a signed graph with an induced negative triangle or its underlying graph $G$ is triangle-free.

If $\Sigma$ has an induced negative triangle, then its spectrum is $\{1,1,-2\}$. By Theorem [2.3, we have

$$
\lambda_{2}(\Sigma) \geq 0
$$

If $\Sigma$ is triangle free signed graph, then it has an induced path on 3 vertices whose spectrum is $\{\sqrt{2}, 0,-\sqrt{2}\}$. Now, by Theorem [2.3, we have

$$
\lambda_{2}(\Sigma) \geq 0
$$

Thus in either of the cases, we have the result.
Proof of Theorem 4.2; Let $\left(n^{+}, n^{-}, n^{0}\right)$ denote the inertia of $\Sigma$, where $n^{+}, n^{-}$and $n^{0}$ are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of $A(\Sigma)$, respectively.

Let $\lambda_{1}(\Sigma) \geq\left|\lambda_{n}(\Sigma)\right|$. Set $S^{+}=\lambda_{1}^{2}+\cdots+\lambda_{n^{+}}^{2}$ and $S^{-}=\lambda_{n-n^{-+1}}^{2}+\cdots+\lambda_{n}^{2}$. Suppose that $\lambda_{1}^{2}(\Sigma)+\lambda_{2}^{2}(\Sigma)>m$. Since $S^{+}+S^{-}=2 m$, we have

$$
\lambda_{1}^{2}(\Sigma)+\lambda_{2}^{2}(\Sigma)>m=\frac{S^{+}+S^{-}}{2}
$$

and we get,

$$
\begin{gathered}
\lambda_{1}^{2}(\Sigma)+\lambda_{2}^{2}(\Sigma) \geq 2\left(\lambda_{1}^{2}(\Sigma)+\lambda_{2}^{2}(\Sigma)\right)-S^{+}>S^{-} \geq 0 \\
\lambda_{1}^{2}(\Sigma)+\lambda_{2}^{2}(\Sigma)>S^{-}
\end{gathered}
$$

Now we construct two $n^{-}$-vectors $x$ and $y$ such that $x=\left(\lambda_{1}^{2}(\Sigma), \lambda_{2}^{2}(\Sigma), 0, \ldots, 0\right)^{T}, y=$ $\left(\lambda_{n}^{2}(\Sigma), \lambda_{n-1}^{2}(\Sigma), \ldots, \lambda_{n-n^{-+1}}^{2}(\Sigma)\right)^{T}$. Since $\lambda_{1}^{2}(\Sigma)+\lambda_{2}^{2}(\Sigma)>S^{-}$, we have $y \prec_{w} x$ and $x \neq y$. Set $p=\frac{3}{2}$, by Theorem [2.2, we have

$$
\|x\|_{3 / 2}^{3 / 2}>\|y\|_{3 / 2}^{3 / 2}
$$

that is,

$$
\lambda_{1}^{3}(\Sigma)+\lambda_{2}^{3}(\Sigma)>\left|\lambda_{n}^{3}(\Sigma)\right|+\left|\lambda_{n-1}^{3}(\Sigma)\right|+\cdots+\left|\lambda_{n-n^{-}+1}^{3}(\Sigma)\right|
$$

Since we know that in a signed graph without balanced triangles,

$$
\sum_{i=1}^{n} \lambda_{i}^{3}(\Sigma)=6 t_{s}(\Sigma)=6\left(t^{+}(\Sigma)-t^{-}(\Sigma)\right)<0
$$

This implies that

$$
6 t_{s}(\Sigma)=\sum_{i=1}^{n} \lambda_{i}^{3}(\Sigma)>\lambda_{1}^{3}(\Sigma)+\lambda_{2}^{3}(\Sigma)+\lambda_{n}^{3}(\Sigma)+\lambda_{n-1}^{3}(\Sigma)+\cdots+\lambda_{n-n^{-}+1}^{3}(\Sigma)>0
$$

This is a contradiction. Thus we have our desired result.
Let $t_{s}(\Sigma)$ denote the number of positive triangles minus the number of negative triangles in $\Sigma$. The technique used in the proof of Theorem4.2 is similar to that of Lin et al. [9]. This technique works if the assumption of balanced triangle free is replaced by the assumption that number of positive triangles is less than the number of negative triangles, that is $t_{s}(\Sigma) \leq 0$. From this we can conclude that in a given signed graph $\Sigma$, either $\lambda_{1}(\Sigma)^{2} \leq m$ or $\lambda_{n}(\Sigma)^{2} \leq m$, as either $t_{s}(\Sigma) \leq 0$ or $t_{s}(-\Sigma) \leq 0$ is always true.

In the following theorem, we give a different bound for the largest eigenvalue of a signed graph in terms of the total number of triangles present in the signed graph whenever the number of positive triangles is greater than the number of negative triangles in the signed graph.

Theorem 4.4. Let $\Sigma$ be a signed graph with $n$ vertices and $m$ edges with $n \geq 3$. Let $\lambda_{1}(\Sigma)$ be the largest eigenvalue of $A(\Sigma)$. If $t_{s}(\Sigma) \geq 0$, then

$$
\lambda_{1}^{2}(\Sigma) \leq m+\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}
$$

Proof. Let $\left(n^{+}, n^{-}, n^{0}\right)$ denote the inertia of $\Sigma$, where $n^{+}, n^{-}$and $n^{0}$ are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of $A(\Sigma)$, respectively.

Let $S^{+}=\lambda_{1}^{2}+\cdots+\lambda_{n^{+}}^{2}$ and $S^{-}=\lambda_{n-n^{-+1}}^{2}+\cdots+\lambda_{n}^{2}$. Suppose that $\lambda_{1}^{2}(\Sigma)>m+\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}$. Since $S^{+}+S^{-}=2 m$, we have

$$
\lambda_{1}^{2}(\Sigma)>m+\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}=\frac{S^{+}+S^{-}}{2}+\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}
$$

which implies that $2\left(\lambda_{1}^{2}(\Sigma)-\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}\right)-S^{+}>S^{-}$. Now, it is easy to see that $\lambda_{1}^{2}(\Sigma)-$

$$
\begin{aligned}
&\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}} \geq 2\left(\lambda_{1}^{2}(\Sigma)-\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}\right)-S^{+} . \text {Thus } \\
& \lambda_{1}^{2}(\Sigma)-\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}>S^{-}
\end{aligned}
$$

Define two $\left(n^{-}+1\right)$-vectors $x$ and $y$ as follows:

$$
x=\left(\lambda_{1}^{2}(\Sigma), 0,0, \ldots, 0\right)^{T}, \quad y=\left(\lambda_{n}^{2}(\Sigma), \lambda_{n-1}^{2}(\Sigma), \ldots, \lambda_{n-n^{-}+1}^{2}(\Sigma),\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}\right)^{T}
$$

Since $\lambda_{1}^{2}(\Sigma)-\left(6 t_{s}(\Sigma)\right)^{\frac{2}{3}}>S^{-}$, we have $y \prec_{w} x$ and $x \neq y$. Set $p=\frac{3}{2}$, by Theorem 2.2, we have

$$
\|x\|_{3 / 2}^{3 / 2}>\|y\|_{3 / 2}^{3 / 2}
$$

that is,

$$
\lambda_{1}^{3}(\Sigma)>\left|\lambda_{n}^{3}(\Sigma)\right|+\left|\lambda_{n-1}^{3}(\Sigma)\right|+\cdots+\left|\lambda_{n-n^{-}+1}^{3}(\Sigma)\right|+6 t_{s}(\Sigma) .
$$

Since we know that in a signed graph $\sum_{i=1}^{n} \lambda_{i}^{3}(\Sigma)=6 t_{s}(\Sigma)$, we get

$$
6 t_{s}(\Sigma)=\sum_{i=1}^{n} \lambda_{i}^{3}(\Sigma)>\lambda_{1}^{3}(\Sigma)+\lambda_{n}^{3}(\Sigma)+\lambda_{n-1}^{3}(\Sigma)+\cdots+\lambda_{n-n^{-}+1}^{3}(\Sigma)>6 t_{s}(\Sigma)
$$

This is a contradiction. Thus we have our desired result.
Remark 4.1. Note that Theorem 4.4, in case of unsigned graphs, simplifies to the inequality

$$
\lambda_{1}^{2}(G) \leq m+(6 t(G))^{\frac{2}{3}}
$$

which is equivalent to Theorem 1.2 in case of triangle free graphs, but it is different when the graph is not triangle free.

## 5 Signed walks and largest eigenvalue

A walk in a signed graph is positive if the number of its negative edges is even (including the repeating edges); otherwise, it is negative. In the same way we decide whether a cycle in a signed graph is positive or negative. Let $w_{k}^{+}(i, j)$ denote the number of positive walks of length $k-1$ starting at $i$ and terminating at $j$. Define $w_{k}^{+}(i)=w_{k}^{+}(i, i)$ for each $1 \leq i \leq n$. Similarly $w_{k}^{-}(i, j)$ and $w_{k}^{-}(i)$ are defined in terms of the negative walks. Let $w_{r}^{+}(\Sigma)$ and $w_{r}^{-}(\Sigma)$ denote the number of positive walks and negative walks in signed graph $\Sigma$ of length $r-1$, respectively.

For a signed graph $\Sigma=(G, \sigma)$, the $(i, j)$-th entry of $A(\Sigma)^{r-1}$ is the difference between the number of positive and negative walks of length $r-1$ between the vertices $v_{i}$ and $v_{j}$ in
$\Sigma$. Hence the expression $e^{T} A(\Sigma)^{r-1} e$ is the difference between the number of positive and negative walks of length $r-1$ in $\Sigma$. Define $w_{k}(\Sigma)=e^{T} A(\Sigma)^{r-1} e$.

The total number of positive and negative walks in $\Sigma$ need not be switching invariant. We define $r$-frustration index of signed graph $\Sigma$, denoted by $\epsilon_{r}(\Sigma)$, to be the minimum number of negative walks of length $r-1$ in the switching equivalence class of $\Sigma$. That is,

$$
\epsilon_{r}(\Sigma)=\min _{\Sigma^{\prime} \sim \Sigma} w_{r}^{-}(\Sigma)
$$

Note that, $\epsilon_{2}(\Sigma)=\epsilon(\Sigma)$, and thus $\epsilon_{r}(\Sigma)$ generalizes the notion of frustration index to signed walks of higher lengths.

Theorem 5.1. Let $\Sigma=(G, \sigma)$ be a signed graph with $n$ vertices and $m$ edges. Let $A(\Sigma)$ be its signed adjacency matrix and $\omega_{b}(\Sigma)$ be its balanced clique number. Let $w_{r}(G)$ denote the total number of walks in $G$ and $\epsilon_{r}(\Sigma)$ be the $r$-frustration index of $\Sigma$. Let $\lambda_{1}(\Sigma)$ be the largest eigenvalue of $A(\Sigma)$. Then

$$
\lambda_{1}^{r}(\Sigma) \leq\left(w_{r}(G)-\epsilon_{r}(\Sigma)\right)\left(1-\frac{1}{\omega_{b}(\Sigma)}\right)
$$

Proof. Let $\Sigma^{\prime}=\left(G, \sigma^{\prime}\right)$ be the signed graph obtained $\Sigma$ by applying the switching function $\eta$ defined in Theorem 3.3. From the equation 3.1, we have $\lambda_{1}(\Sigma) \leq \sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j}$. Let $\Sigma_{+}^{\prime}$ be the signed subgraph of $\Sigma^{\prime}$ spanned by the set of edges $\sigma_{i j}^{\prime}=1$. Let the number of $r$-walks in the all positive signed subgraph $\Sigma_{+}^{\prime}$ be $w_{r}\left(\Sigma_{+}^{\prime}\right)$ and its clique number be $\omega\left(\Sigma_{+}^{\prime}\right)$. Then $\lambda_{1}(\Sigma) \leq \sum_{i \sim j, \sigma_{i j}^{\prime}=1} 2 y_{i} y_{j} \leq \lambda_{1}\left(\Sigma_{+}^{\prime}\right)$. By Theorem 1.5, we have

$$
\lambda_{1}^{r}(\Sigma) \leq \lambda_{1}^{r}\left(\Sigma_{+}^{\prime}\right) \leq w_{r}\left(\Sigma_{+}^{\prime}\right)\left(1-\frac{1}{\omega\left(\Sigma_{+}^{\prime}\right)}\right) .
$$

Since $\Sigma_{+}^{\prime}$ is all positive signed subgraph of $\Sigma^{\prime}$, we have $w_{r}\left(\Sigma_{+}^{\prime}\right) \leq w_{r}(G)-\epsilon_{r}(\Sigma)$ and $\omega_{+} \leq$ $\omega_{b}(\Sigma)$. Thus

$$
\lambda_{1}^{r}(\Sigma) \leq\left(w_{r}(G)-\epsilon_{r}(\Sigma)\right)\left(1-\frac{1}{\omega_{b}(\Sigma)}\right) .
$$

Note when $r=1$, we obtain Theorem 1.4 of Wang et al [16].
Our next objective is to extend the left-hand side of the inequality of Theorem 1.5, Before that, we give a theorem that will be helpful.

Theorem 5.2. Let $\Sigma=(G, \sigma)$ be a signed graph of order $n$ with eigenvalues $\lambda_{1}(\Sigma) \geq$ $\lambda_{2}(\Sigma) \geq \cdots \geq \lambda_{n}(\Sigma)$ and $u_{1}, \ldots, u_{n}$ be the corresponding orthogonal unit eigenvectors. For every $1 \leq i \leq n$, let $u_{i}=\left(u_{i 1}, \ldots, u_{i n}\right)$ and $c_{i}=\left(\sum_{j=1}^{n} u_{i j}\right)^{2}$. Then

$$
w_{k}(\Sigma)=\sum_{i=1}^{n} c_{i} \lambda_{i}^{k-1}(\Sigma)
$$

Proof. Since $A(\Sigma)$ is a symmetric matrix, by the definition of $w_{k}(\Sigma)$ and the spectral theorem, we have

$$
w_{k}(\Sigma)=e^{T} A(\Sigma)^{k-1} e=(U e)^{T} D(\Sigma)(U e)
$$

It is easy to see that, $U e=\left(\sum_{j=1}^{n} u_{1 j}, \ldots, \sum_{j=1}^{n} u_{n j}\right)^{T}$, and thus the result follows.
Theorem 5.3. Let $\Sigma=(G, \sigma)$ be a signed graph with $n$ vertices and $m$ edges. Let $A(\Sigma)$ be its signed adjacency matrix. Let $w_{r}(\Sigma)$ denote the number of signed r-walks in $\Sigma$. Let $\lambda_{1}(\Sigma) \geq \lambda_{2}(\Sigma) \geq \cdots \geq \lambda_{n}(\Sigma)$ be the eigenvalues of the adjacency matrix of $\Sigma$. Let $\rho(\Sigma)$ be the spectral radius of $A(\Sigma)$. Then for all $r>0$ and odd $q>0$,

$$
\frac{w_{q+r}(\Sigma)}{w_{q}(\Sigma)} \leq \rho^{r}(\Sigma)
$$

Proof. From Theorem 5.2, we have

$$
\begin{aligned}
\frac{w_{q+r}(\Sigma)}{w_{q}(\Sigma)} & =\frac{\sum_{i=1}^{n} c_{i} \lambda_{i}^{q+r-1}(\Sigma)}{\sum_{i=1}^{n} c_{i} \lambda_{i}^{q-1}(\Sigma)} \\
& =\rho^{r}(\Sigma) \frac{\sum_{i=1}^{n} c_{i}\left(\frac{\lambda_{i}(\Sigma)}{\rho(\Sigma)}\right)^{q+r-1}}{\sum_{i=1}^{n} c_{i}\left(\frac{\lambda_{i}(\Sigma)}{\rho(\Sigma)}\right)^{q-1}} \\
& \leq \rho^{r}(\Sigma) .
\end{aligned}
$$

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