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For Matrix Recovery, Rank Restricted Isometry Property and Robust Uniform Boundedness Property Imply Rank Robust Null Space Property

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***Abstract*— Compressed sensing refers to the recovery of high-dimensional but low-complexity objects from a small number of measurements. The recovery of sparse vectors and the recovery of low-rank matrices are the main applications of compressed sensing theory. In vector recovery, the restricted isometry property (RIP) and the robust null space property (RNSP) are the two widely used sufficient conditions for achieving compressed sensing. Until recently, RIP and RNSP were viewed as two separate sufficient conditions. However, in a recent paper [1], the present authors have shown that in fact the RIP implies the RNSP, thus establishing the fact that RNSP is a weaker sufficient condition than RIP.**

In matrix recovery, there are three different sufficient conditions for achieving low-rank matrix reconstruction, namely; Rank Restricted Isometry Property (RRIP), Rank Robust Null Space Property (RRNSP), and Robust Uniform Boundedness Property (RUBP). In this paper, using the result of [1], it is shown that actually both RRIP and RUBP imply the RRNSP, so that RRNSP is the weakest sufficient condition for matrix recovery. In contrast with the situation for vector recovery, until now there are no deterministic methods for designing a measurement operator for matrix recovery. The present results open the door towards such a possibility.

I. Introduction

A. Overview

In many signal processing applications the data to be processed can be represented as a matrix with real valued entries. It is often natural or reasonable to assume that the observed data is a low-rank matrix with some added noise. In this paper, we focus on the problem of recovering an unknown low-rank $n_r \times n_c$ matrix X from its noisy linear measurement via nuclear norm minimization.

At present there are three different sufficient conditions for matrix recovery, known as **Rank Restricted Isometry Property (RRIP)**, **Rank Robust Null Space Property (RRNSP)**, and **Robust Uniform Boundedness Property (RUBP)**. In this work, we study the relation between these three properties. We show that under the appropriate conditions, both the RRIP and RUBP imply RRNSP which establishes the fact that RRNSP is the weakest sufficient condition for matrix recovery. These results draw upon another paper [1] on vector recovery by the present authors, in which it is shown that the Restricted Isometry Property implies the robust null space property.

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B. Problem Formulation

If $X \in \mathbb{R}^{n_r \times n_c}$, the symbols $\|X\|_N$, $\|X\|_F$ and $\|X\|_S$ denote respectively the nuclear norm, the Frobenius norm, and the spectral norm of X. Specifically, if $\boldsymbol{\sigma}(X)$ denotes the vector of singular values of X, then $\|X\|_N$, $\|X\|_F$ and $\|X\|_S$ equal the ℓ_1-norm, the ℓ_2-norm, and the ℓ_∞ of $\boldsymbol{\sigma}(X)$, respectively. Let $\mathcal{M}(k)$ denote the set of all matrices in $\mathbb{R}^{n_r \times n_c}$ of rank k or less. Define the quantities

$$\begin{aligned} \theta_k(X, \|\cdot\|_N) &:= \arg\min_{Z \in \mathcal{M}(k)} \|X - Z\|_N = \sum_{i=k+1}^{n_1} \sigma_i, \\ \bar{\theta}_k(X, \|\cdot\|_N) &:= \sum_{i=1}^{k} \sigma_i. \end{aligned}$$

Now we are in a position to define matrix recovery problem formally. The objective of compressed sensing is to find, if possible, an integer m (the number of measurements), a linear "measurement" map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$, and another "decoder" map $\Delta : \mathbb{R}^m \to \mathbb{R}^{n_r \times n_c}$ such that the following properties hold:

1) If the measurement y equals $\mathcal{A}(X)$ (i.e., noiseless measurements), and X has rank $\leq k$, then
$$\Delta(y) = X, \ \forall X \in \mathcal{M}(k). \tag{1}$$
This is known as **exact recovery of rank** k.
2) More generally, there exists a constant C such that
$$\|\Delta(y) - X\|_N \leq C\theta_k(X), \ \forall X \in \mathbb{R}^{n_r \times n_c}. \tag{2}$$
This is known as **stable recovery of rank** k.
3) If the measurement y equals $\mathcal{A}(X)+\eta$ where $\|\eta\|_2 \leq \epsilon$ (i.e., noisy measurements with a known upper bound on the noise), then there exist constants C, D such that
$$\|\Delta(y) - X\|_N \leq C\theta_k(X) + D\epsilon, \ \forall X \in \mathbb{R}^{n_r \times n_c}. \tag{3}$$
This is known as **robust recovery of rank** k.

Note that *any* linear map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is of the form

$$\mathcal{A}(X) = \begin{bmatrix} \langle A_1, X \rangle_F \\ \vdots \\ \langle A_m, X \rangle_F \end{bmatrix}, \tag{4}$$

where $A_1, \ldots, A_m \in \mathbb{R}^{n_r \times n_c}$ and $\langle \cdot, \cdot \rangle_F$ denotes the Frobenius inner product, that is,

$$\langle A, B \rangle_F = \text{tr}(AB^\top) = \sum_{i,j} A_{ij} B_{ij}. \quad (5)$$

Clearly robust recovery implies stable recovery, which in turn implies exact recovery. In a way, robust rank recovery ensures robust reconstruction against the noise and stable reconstruction with respect to approximately low- rank matrices. Therefore, one would like to have such a pair $(\mathcal{A}, \Delta)$ that achieves robust rank recovery.

The constrained nuclear norm minimization is the widely used decoder map in low-rank matrix reconstruction, i.e.,

$$\hat{X} = \Delta_N(y) := \underset{Z \in \mathbb{R}^{n_r \times n_c}}{\arg\min} \|Z\|_N \text{ s.t. } \|\mathcal{A}(Z) - y\|_2 \leq \epsilon. \quad (6)$$

With the decoder map Δ_N, the challenge is to find a measurement map $\mathcal{A}$, such that the pair $(\mathcal{A}, \Delta_N)$ achieves robust rank recovery.

II. Literature Review

There are many results for the recovery of sparse vectors, starting with the work of Candès and coworkers, and Donoho and coworkers. Rather than give an exhaustive bibliography, we refer the reader to the comprehensive text [2] and the references therein. It suffices to say that the most popular approach is ℓ_1-norm minimization, that is,

$$\Delta(y) := \arg\min_z \|z\|_1 \text{ s.t. } \|Az - y\|_2 \leq \epsilon. \quad (7)$$

The matrix A can be chosen to satisfy either the **Restricted Isometry Property (RIP)**, or the **Robut Null Space Property (RNSP)**. These properties are defined respectively in [2, Definition 4.17] and [2, Chapter 6]. Until recently, the relationship between the two properties was not very clear. However, a recent paper by the present authors [1] shows that in fact the RIP implies the RNSP.

In contrast with vector recovery, there are relatively few results on matrix recovery. One of the first sufficient condition for low-rank matrix is known as the RRIP, defined next.

Definition 1: A linear map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is said to satisfy the **Rank Restricted Isometry Property (RRIP)** of rank k with constant δ_k, if

$$(1-\delta_k)\|X\|_F^2 \leq \|\mathcal{A}(X)\|_F^2 \leq (1+\delta_k)\|X\|_F^2, \ \forall X \in \mathcal{M}(k). \quad (8)$$

In one of the earliest results in matrix recovery in [3], it is shown that if each measurement matrix A_i in $\mathcal{A}$ consists of $n_r n_c$ random samples of a normal Gaussian variable, then such a map satisfies the RRIP of rank k for a suitably defined constant δ_k, with probability close to one. In turn this result is used to show that the decoder map Δ_N in (6) achieves robust recovery of rank k, under appropriate conditions.

The rank robust null space property is introduced as an exercise in [2, Problem 4.2].

Definition 2: A linear map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is said to satisfy the **Rank Robust Null Space Property (RRNSP)** of rank k if there exists a constant $\rho \in (0, 1)$ and another constant $\tau \geq 0$ such that every matrix $X \in \mathbb{R}^{n_r \times n_c}$ satisfies

$$\sum_{i=1}^{k} \sigma_i(X) \leq \rho \sum_{i=k+1}^{n} \sigma_i(X) + \tau \|\mathcal{A}(X)\|_2, \ \forall X \in \mathbb{R}^{n_r \times n_c}. \quad (9)$$

In [2, Problem 4.2], the reader is asked to show that the RRNSP of rank k is sufficient to ensure that the decoder Δ_N in (6) achieves robust recovery of rank k.

Yet a third sufficient condition for robust recovery of rank k is presented in [4]. The premise of [4] is that taking m different Frobenius inner products as in (4) can be very time-consuming. Instead it is suggested to choose matrix A_i to be a *rank one* matrix of the form $bc^\top$, because, as is easily verified, the Frobenius inner product $\langle bc^\top, X \rangle_F$ equals the triple product $b^\top X c$.

Definition 3: A linear measurement map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ $(n_r \leq n_c)$ is said to satisfy the **Robust Uniform Boundedness Property (RUBP)** of order $r \leq n_r$ if for all $X \in \mathcal{M}(r)$, it is true that

$$C_1 \leq \frac{\|\mathcal{A}(X)\|_1 / m}{\|X\|_F} \leq C_2, \quad (10)$$

where $C_1, \ C_2$ are some positive constants.

It is shown in [4] that, if vectors $b_i, c_i, i = 1, \ldots, m$ are chosen to be random samples of a normal Gaussian, then the resulting linear map $\mathcal{A}$ satisfies the RUBP.

It is further shown in [4] that under the appropriate conditions, RUBP enables the pair $(\mathcal{A}, \Delta_N)$ to achieve robust rank recovery.

III. Our Contribution

As mentioned above, until now there are three different sets of conditions on the measurement map $\mathcal{A}$, namely: RRIP, RRNSP and RUBP. All these properties (together with the decoder (6)) guarantee robust recovery of rank k. Now, in the present paper, we prove that both RRIP and RUBP imply RRNSP.

A. RRIP implies the RRNSP

In this sub-section we show that the Rank Restricted Isometry Property (RRIP) implies the Rank Robust Null Space Property (RRNSP). Because our proof draws upon a similar result for vector recovery, namely that RIP implies RNSP, we refer the reader to [1] for details. Now, to facilitate the statement of our theorem, we introduce some notation. Suppose $t > 1$. Define

$$v := \sqrt{t(t-1)} - (t-1) \in (0, 0.5)$$

$$a := [v(1-v) - \delta(0.5 - v + v^2)]^{1/2}, \quad (11)$$

$$b := v(1-v)\sqrt{1+\delta}, \quad (12)$$

$$c := \left[\frac{\delta v^2}{2(t-1)} \right]^{1/2}. \quad (13)$$

Then we have the following:

Theorem 1: Suppose $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ satisfies RRIP of rank tk with $\delta_{tk} < \sqrt{(t-1)/t}$ for some $t > 1$. Define the constants a, b, c as in (11),(12),(13). Then, $\mathcal{A}$ satisfies rank robust null space property of rank k with

$$\rho := c/a, \ \tau := b\frac{\sqrt{k}}{a^2}.$$

B. RUBP implies the RRNSP

In this sub-section we show that under some suitable condition RUBP implies RRNSP. This final result establishes the fact that RRNSP is the weakest sufficient condition. Now we present our main result.

Theorem 2: Suppose that for some $t \geq 2$, $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ satisfies RUB of order tk with constants C_1, C_2 such that $C_2/C_1 < \sqrt{t}$. Then $\mathcal{A}$ satisfies RRNSP of rank k with

$$\rho := \frac{C_2/C_1}{\sqrt{t}}, \ \tau := \frac{\sqrt{k/m}}{C_1}.$$

IV. Conclusion

In this paper we have studied the problem of matrix recovery via nuclear norm minimization. We have studied the three currently known sufficient conditions, namely Rank Restricted Isometry Property (RRIP), Rank Robust Null Space Property (RRNSP), and Robust Uniform Boundedness Property (RUBP). In this paper, it is shown that actually both RRIP and RUBP imply the RRNSP, so that RRNSP is the weakest sufficient condition for matrix recovery. These results draw upon another paper on vector recovery by the present authors [1], in which it is shown that the Restricted Isometry Property implies the robust null space property. In contrast with the situation for vector recovery, until now there are no deterministic methods for designing a measurement operator for matrix recovery. The present results open the door towards such a possibility.

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